

Symmetry Properties of the k -Body Embedded Unitary Gaussian Ensemble of Random Matrices

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Abstract

We extend the recent study of the k -body embedded Gaussian ensembles by Benet *et al.* (Phys. Rev. Lett. 87 (2001) 101601-1 and Ann. Phys. 292 (2001) 67) and by Asaga *et al.* (cond-mat/0107363 and cond-mat/0107364). We show that central results of these papers can be derived directly from the symmetry properties of both, the many-particle states and the random k -body interaction. We offer new insight into the structure of the matrix of second moments of the embedded ensemble, and of the supersymmetry approach. We extend

the concept of the embedded ensemble and define it purely group-theoretically.

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1 Introduction

Recently, novel results on the spectral properties of the k -body embedded unitary and orthogonal ensembles of random matrices were obtained both for Fermions (Refs. [1] and [2]) and for Bosons (Refs. [3] and [4]). We refer to these two pairs of papers as to ABRW. These results were based on explicit analytical calculations. The authors did mention that their work had its root in the underlying symmetry properties of the embedded ensembles. They did not, however, display these symmetry properties explicitly, nor did they derive their results with the help of such properties.

In the present paper, we remedy this situation. We show that central results of ABRW follow from the symmetry properties of the many-particle states and of the random k -body interaction. For simplicity, we confine ourselves to the k -body embedded unitary ensemble $\text{EGUE}(k)$ although our results can certainly be generalized to the orthogonal case.

To define $\text{EGUE}(k)$, we consider m particles (Fermions or Bosons) in l degenerate single-particle states. The random k -body interaction connects states in the Hilbert space of these m particles. It is useful to introduce, for arbitrary t , the dimension N_t of the Hilbert space of t particles. For Fermions, we have $N_t = \binom{l}{t}$ while for Bosons, $N_t = \binom{l+t-1}{t}$. For simplicity, we write N for N_m . We show that for the embedded ensemble, three symmetry groups are relevant. These are (i) the group $\text{SU}(l)$ of unitary transformations of the degenerate l single-particle states; (ii) the group $\text{U}(N_k)$ of unitary transformations of the k -body interaction; (iii) the group $\text{U}(N)$ of unitary transformations of the Hilbert space. The group $\text{SU}(l)$ governs the embedding. The group $\text{U}(N_k)$ is the symmetry group of $\text{EGUE}(k)$. The group $\text{U}(N)$ is the symmetry group of the Gaussian unitary ensemble (GUE), i.e., the free Gaussian unitary ensemble in N dimensions.

The matrix elements of the k -body interaction in the space of m -particle states are Gaussian random variables with zero mean. Therefore, all information necessary for calculating averages over $\text{EGUE}(k)$ is contained in the matrix of second moments of the ensemble. We show that the spectral representation (the “eigenvalue expansion” of ABRW) of the matrix of second moments constructed explicitly in ABRW can also be obtained from symmetry arguments. To this end, we decompose the random k -body interaction into a sum of k -particle basic interaction terms which transform according to the irreducible representations of $\text{SU}(l)$. The latter are specified

by the bodyness (unitary rank) quantum number b introduced by Mon and French [5]. The eigenvalues appearing in the spectral representation of the matrix of second moments are uniquely defined by this quantum number, and the eigenvectors are given by $SU(l)$ Clebsch–Gordan coefficients.

We isolate the $U(N)$ –invariant part of the matrix of second moments. The result is unexpected as this part consists of two pieces. We find the expected term which has the form of the matrix of second moments of the free GUE, or, equivalently, of $EGUE(m)$. However, we also find another term which has the form of the matrix of second moments of the degenerate Gaussian ensemble of multiples of the unit matrix, or, equivalently, of $EGUE(0)$. We show that for $k \ll m$, the $U(N)$ –invariant part favors a Gaussian spectral shape and non–ergodic spectral properties. For $2k > m$, on the other hand, that part favors a semicircular spectral shape, and ergodic behavior.

We also investigate the invariance of the supersymmetric n –point generating functions of the ensemble under the unitary transformations of the interactions, and we examine the consequences of that symmetry for the saddle–point solutions. Finally, we extend the concept of the embedded ensemble to cases where the embedding is governed by an arbitrary compact group different from $SU(l)$.

The paper is organized in the following way. Some basic relations for the representations of $SU(l)$ in Fock space are presented in Section 2. The ensemble $EGUE(k)$, its symmetry and the $SU(l)$ expansion of the embedded interactions are discussed in Section 3. The symmetry properties of the matrix of second moments are discussed in Section 4. In Section 5, we isolate the $U(N)$ –invariant part of the matrix of second moments and determine its relative importance. The properties of the supersymmetric generating functions are investigated in Section 6. Comments on the application of the ABRW approach to other ensembles and concluding remarks are contained in Section 7. Some mathematical details, including the evaluation of eigenvalues of the matrix of second moments, are given in Appendices. We always treat Fermions and Bosons jointly. For the representation theory we follow the books by Wybourne [6] and Lichtenberg [7].

2 Fock Space and the Group $SU(l)$

In this Section, we introduce the group $SU(l)$ and describe the action of group operations on composite operators. We use a running label $i = 1, \dots, l$ for the degenerate single-particle states. We use second quantization and denote the single-particle creation and annihilation operators by a_i^+ and a_i , respectively. The vacuum state is written as $|\rangle$. We consider both Fermions and Bosons. Hilbert space is spanned by $N = N_m$ orthonormal m -particle states denoted by $|m v_m\rangle$. The index $v_m = 1, \dots, N$ is a running label. We emphasize that the basis states $|m v_m\rangle$ are not necessarily simple states (Slater determinants for Fermions or product states for Bosons as used in ABRW): Linear combinations of such states are also admitted. No specific assumption on the choice of the basis will be made. The creation operators for the basis states are denoted by $A^+(m v_m)$.

An element $u \in SU(l)$ generates a unitary transformation of the single-particle states and a corresponding transformation

$$T(u)a_i^+T^+(u) = \sum_{i'} u_{i'i} a_{i'}^+, \quad u \in SU(l), \quad (1)$$

of the creation operators a_i^+ . The operators $T(u)$ are generated by the traceless linear forms of $a_i^+ a_{i'}$. The transformation u induces a linear transformation of the operators $A^+(m v_m)$. These operators transform according to the irreducible representations $D^{f_m}(u)$,

$$T(u)A^+(m v_m)T^+(u) = \sum_{v'_m} D_{v'_m v_m}^{f_m}(u) A^+(m v'_m). \quad (2)$$

In the Fermionic (Bosonic) case, the matrices $D^{f_m}(u)$ are given, to within an equivalence transformation, by the totally antisymmetrized (totally symmetrized, respectively) powers of u . In the sequel, we write the Young tableaux of the irreducible representations of $SU(l)$ as $h = (h_1, \dots, h_{l-1})$, with h_j denoting the number of columns of length j . The symbol 0^j stands for j zeros. In the case of Fermions, $D^{f_m}(u)$ belongs to the Young tableau $f_m = (0^{m-1} 1 0^{l-m-1})$; in the case of Bosons, it belongs to the Young tableau $f_m = (m 0^{l-2})$. The annihilation operators $A(m v_m)$ transform according to the conjugate irreducible representation $\mathcal{D}^{\bar{f}_m}(u) = (D^{f_m}(u))^*$, where $\bar{f}_m = (0^{l-m-1} 1 0^{m-1})$ for Fermions, and $\bar{f}_m = (0^{l-2} m)$ for Bosons.

In analogy with the familiar fractional–parentage technique of first quantization, we can expand the m –particle states $|mv_m\rangle$ into a sum of products of the k –particle creation operators $A^+(kv_k)$ and the $(m-k)$ –particle states $|m-k\ v_{m-k}\rangle$. Explicitly,

$$|mv_m\rangle = \binom{m}{k}^{-1/2} \sum_{v_k v_s} A^+(kv_k) |sv_s\rangle C_{f_k v_k f_s v_s}^{f_m v_m}. \quad (3)$$

Because of the duality relation, Eq. (27) below, the particle rank k and the particle rank $m-k$ are intimately linked. This is why we use here and in the sequel the label

$$s = m - k. \quad (4)$$

The coefficients $C_{f_k v_k f_s v_s}^{f_m v_m}$ are the $SU(l)$ Clebsch–Gordan coefficients for the coupling of basic states of the representations $D^{f_k}(u)$ and $D^{f_s}(u)$ to the basic states of the representation $D^{f_m}(u)$ (in short, the Clebsch–Gordan coefficients for the coupling $(f_k f_s) f_m$). In the fractional–parentage terminology, these $SU(l)$ Clebsch–Gordan coefficients represent the k –particle coefficients of fractional parentage of the m –particle states, $C_{f_k v_k f_s v_s}^{f_m v_m} = [kv_k sv_s] \{mv_m\}$. The origin of the combinatorial factor in Eq. (3) and further details are explained in Appendix A.

The pair $A^+(kv_k)A(kv'_k)$ of k –particle interaction operators transforms according to the direct product of the irreducible $SU(l)$ representations $D^{f_k}(u)$ and $\mathcal{D}^{\bar{f}_k}(u)$. With the help of an appropriate Clebsch–Gordan transformation, this product can be reduced to a direct sum

$$D^{f_k}(u) \times \mathcal{D}^{\bar{f}_k}(u) = \sum_b D^{g_b}(u) \quad (5)$$

of irreducible representations $D^{g_b}(u)$ of Young tableaux g_b . These are uniquely specified by the “bodyness” quantum number [5] b which assumes the values $b = 0, \dots, k$. For Fermions, we have $g_b = (0^{b-1} 10^{l-2b-1} 10^{b-1})$ while for Bosons, $g_b = (b 0^{l-3} b)$. (In keeping with ABRW, we confine ourselves for Fermions to the case of less than half filling, $2m \leq l$.) The set of basic k –particle interactions $B_k(bw_b)$ which transform according to the irreducible representations $D^{g_b}(u)$ of $SU(l)$,

$$T(u) B_k(bw_b) T^+(u) = \sum_{w'_b} D_{w'_b w_b}^{g_b}(u) B_k(bw'_b), \quad (6)$$

is given by

$$B_k(bw_b) = \sum_{v_k v'_k} A^+(kv_k) A(kv'_k) C_{f_k v_k \bar{f}_k v'_k}^{g_b w_b} . \quad (7)$$

The $SU(l)$ Clebsch–Gordan coefficients $C_{f_k v_k \bar{f}_k v'_k}^{g_b w_b}$ accomplish the reduction. The indices w_b label the individual states (rows) of the $SU(l)$ representation $D^{g_b}(u)$ of Young tableau g_b . These running indices range from one to $M_b = N_b^2 - N_{b-1}^2$, the dimension of the representation $D^{g_b}(u)$. The representations of the Young tableaux g_b are self-conjugate and integer. Therefore, we can assume that the $D^{g_b}(u)$ are real and that the $B_k(bw_b)$ are Hermitean, $B_k^+(bw_b) = B_k(bw_b)$. No other assumptions about the choice of the irreducible representations $D^{g_b}(u)$ are made.

Eq. (7) implies the relations

$$\langle B_k(bw_b) \rangle_k = \delta_{b0} \sqrt{N_k} , \quad \langle B_k(bw_b) B_k(b'w'_{b'}) \rangle_k = \delta_{bb'} \delta_{w_b w'_{b'}} . \quad (8)$$

The symbol $\langle O \rangle_t$ denotes the trace of O in the space of t -particle states. According to the Wigner-Eckart theorem for the group $SU(l)$, the matrix elements of the operators $B_k(bw_b)$ with respect to the m -particle states $|m v_m\rangle$ are products of the reduced matrix elements $\langle m || B_k(b) || m \rangle$ and of the Clebsch-Gordan coefficients $C_{f_m v_m \bar{f}_m v'_m}^{g_b w_b}$ of $SU(l)$ for the reduction of the direct product $D^{f_m}(u) \times \mathcal{D}^{\bar{f}_m}(u)$ into the direct sum of the representations $D^{g_b}(u)$,

$$\langle m v_m | B_k(bw_b) | m v'_m \rangle = \langle m || B_k(b) || m \rangle C_{f_m v_m \bar{f}_m v'_m}^{g_b w_b} . \quad (9)$$

The definition (9) of the reduced matrix elements generalizes the canonical definition of the reduced matrix elements of the irreducible tensor operators of $SU(2)$ given by Racah [8] to the present case. The dimension-dependent factors $M_b^{-1/2}$ are absorbed in the reduced elements. For further details on the Wigner-Eckart theorem for $SU(l)$, we refer the reader to the book by Lichtenberg [7] and the references given therein.

The norm $\langle B_k^2(bw_b) \rangle_m$ is independent of w_b and equal to the square of the reduced element. For later use we note that the norm of any properly normalized k -particle operator $S_k(b)$ of bodyness b has this same value: Making use of Eq. (9), we find that every linear form $S_k(b)$ of $B_k(bw_b)$ which satisfies the normalization condition $\langle S_k^+(b) S_k(b) \rangle_k = 1$, obeys

$$\langle S_k^+(b) S_k(b) \rangle_m = \langle m || B_k(b) || m \rangle^2 . \quad (10)$$

The matrices $\langle mv_m | B_k(bw_b) | mv'_m \rangle$ play an important role in the sequel. Therefore, it may be useful to give a physical interpretation of the concept of bodyness, and to establish the relation between these matrices and the corresponding quantities appearing in ABRW. As an example, we consider the simplest version of a pair of creation and annihilation operators: We take $k = 1$, i.e., we consider the pair $a_i^\dagger a_j$. For $i \neq j$, this operator has bodyness $b = 1$: It moves one particle from the single-particle state j to a different single-particle state i . For $i = j$, on the other hand, the operator can be written as the sum of two terms with $b = 0$ and $b = 1$, respectively. Indeed, we can decompose $a_i^\dagger a_i$ into a traceless part $a_i^\dagger a_i - (1/l) \sum_{j=1}^l a_j^\dagger a_j$ and the remainder, $(1/l) \sum_{j=1}^l a_j^\dagger a_j$. By virtue of Eq. (8), the traceless part has bodyness $b = 1$, while the remainder has bodyness $b = 0$. By construction, the interaction terms of bodyness $b = k$ cannot be simulated by an interaction of lower particle rank and represent the genuine k -body interaction which generically changes the state of not less than k particles. The example just considered suggests, and a more detailed consideration shows, that the matrices $\langle mv_m | B_k(bw_b) | mv'_m \rangle$ bear a close relationship with the Hermitean matrices $C_{\mu\nu}^{sa}$ of ABRW. ‘Indeed, the sets of indices (bw_b) and (sa) can be identified because the dimensions of both sets are the same and given by M_b . The differences between the matrices $\langle mv_m | B_k(bw_b) | mv'_m \rangle$ and the matrices $C_{\mu\nu}^{sa}$ of ABRW are: (i) We admit any basis and drop the specialization to Slater determinants or product states employed by ABRW. (ii) The normalization condition imposed by ABRW on the matrices $C_{\mu\nu}^{sa}$ differs from our Eq. (8). (iii) The matrices $\langle mv_m | B_k(bw_b) | mv'_m \rangle$ are introduced group-theoretically, while the matrices $C_{\mu\nu}^{sa}$ were constructed explicitly. We return to this comparison in Section 4 below.

3 The Embedded Ensemble

After the preliminary steps of Section 2, we turn to a group-theoretical classification of the embedded ensemble EGUE(k). This ensemble describes m identical particles distributed over l degenerate single-particle states which interact through a random k -particle interaction of GUE type,

$$W(k) = \sum_{v_k v'_k} W_{v_k v'_k}(k) A^+(kv_k) A(kv'_k) . \quad (11)$$

The coefficients $W_{v_k v'_k}(k)$ are N_k^2 independent Gaussian random variables $W_{v_k v'_k}(k) = (W_{v'_k v_k}(k))^*$ with moments

$$\overline{W_{v_k v'_k}(k)} = 0, \quad \overline{W_{v_k v'_k}(k) W_{\tilde{v}_k \tilde{v}'_k}(k)} = \frac{\lambda^2}{N_k} \delta_{v_k \tilde{v}_k} \delta_{v'_k \tilde{v}'_k}. \quad (12)$$

The overbar denotes the average over the ensemble. Obviously, we must have $m \geq k$.

The analysis of EGUE(k) is simplified when we express the interaction $W(k)$ in terms of the operators $B_k(b w_b)$ introduced in Eq. (7). For brevity we write $\kappa = b w_b$, with $\kappa = 0$ referring to the case $b = 0$. Making use of the orthonormality of the traces $\langle B_k(\kappa) B_k(\kappa') \rangle_k$ (cf. Eq. (8)), we find

$$W(k) = \sum_{\kappa} B_k(\kappa) W_k(\kappa), \quad W_k(\kappa) = \langle W(k) B_k(\kappa) \rangle_k. \quad (13)$$

This expansion decomposes the interaction $W(k)$ into parts of well-defined bodyness b with $b = 0, \dots, k$. For l very large compared to m , the terms of highest bodyness $b = k$ dominate and yield the main contribution to the average norm $\overline{\langle W^2(k) \rangle}_m$.

The matrix representation of the ensemble in the m -particle space has the form

$$H_{v_m v'_m}(k) = \langle m v_m | W(k) | m v'_m \rangle. \quad (14)$$

The dimension of the matrices $H(k)$ is equal to the dimension $N = N_m$ of the m -particle Hilbert space. We recall that we do not make any assumption about the basis except that the states $|m v_m\rangle$ are orthonormal. An explicit expression for the matrices $H(k)$ can be found with the help of the fractional-parentage expansion Eq. (3) of the m -particle states. With $s = m - k$, we find from Eq. (11) (for more details we refer to Appendix A)

$$H_{v_n v'_n}(k) = \binom{m}{k} \sum_{v_k v'_k v_s} (C_{f_k v_k f_s v_s}^{f_m v_m})^* C_{f_k v'_k f_s v_s}^{f_m v'_m} W_{v_k v'_k}(k). \quad (15)$$

The dependence of the matrix elements $H_{v_m v'_m}(k)$ of $H(k)$ on the interaction matrix elements $W_{v_k v'_k}(k)$ is, thus, governed by the $SU(l)$ Clebsch-Gordan coefficients $C_{f_k v_k f_s v_s}^{f_m v_m}$ for the coupling $(f_k f_s) f_m$. The expansion Eq. (13) introduces another parametrization of $W(k)$ in terms of the coefficients $W_k(\kappa)$.

The new parameters $W_k(\kappa)$ are real and, by Eqs. (12) and (13), are independent Gaussian random variables with moments

$$\overline{W_k(\kappa)} = 0, \quad \overline{W_k(\kappa)W_k(\kappa')} = \frac{\lambda^2}{N_k} \delta_{\kappa\kappa'} . \quad (16)$$

In the new parametrization, $H_{v_n v'_n}(k)$ is given by

$$H_{v_m v'_m}(k) = \sum_{\kappa} \langle m v_m | B_k(\kappa) | m v'_m \rangle W_k(\kappa) . \quad (17)$$

The average of any function $F(H(k))$ of the Hamiltonian $H(k)$ over the ensemble is given by the integral

$$\overline{F(H(k))} = \int d\mu(W(k)) P(W(k)) F(H(k)) , \quad (18)$$

where $d\mu(W(k))$ denotes the product of differentials of the matrix elements $W_{v_k v'_k}(k)$, and $P(W(k))$ the probability density

$$P(W(k)) = P_0 \exp \left\{ -\frac{N_k}{2\lambda^2} \langle W^2(k) \rangle_k \right\} , \quad P_0 = \left(\frac{N_k}{2\pi\lambda^2} \right)^{N_k^2/2} . \quad (19)$$

In the parametrization of Eq. (17), $\langle W^2(k) \rangle_k = \sum_{\kappa} W_k^2(\kappa)$, and the measure $d\mu(W(k))$ simplifies to the product of differentials of $W_k(\kappa)$.

The free Gaussian unitary ensemble in N dimensions (the GUE) is invariant under unitary transformations. More precisely, let U denote an arbitrary unitary matrix of dimension N , $U \in \text{U}(N)$. Then, with H a GUE Hamiltonian, $\hat{H} = U H U^+$ is also a member of the ensemble and appears in the ensemble with the same weight as the Hamiltonian H . EGUE(k) does not have this symmetry. However, EGUE(k) does possess the $U(N_k)$ symmetry of the Gaussian unitary ensemble for k particles. Indeed, EGUE(k) is invariant under the unitary transformation

$$\hat{W}_{v_k v'_k}(k) = \sum_{\tilde{v}_k \tilde{v}'_k} U_{v_k \tilde{v}_k} W_{\tilde{v}_k \tilde{v}'_k}(k) (U^+)_{\tilde{v}'_k v_k} \quad (20)$$

of the interaction matrices $W_{v_k v'_k}(k)$ by any $N_k \times N_k$ unitary matrix $U \in \text{U}(N_k)$. By this symmetry transformation, the Hamiltonian $H(k)$ containing the coefficients $W(k)$ is replaced by the Hamiltonian $\hat{H}(k)$ containing the coefficients $\hat{W}(k)$. The ensemble average remains invariant, $\overline{F(H(k))} = \overline{F(\hat{H}(k))}$.

For the special matrices $U = D^{f_k}(u)$ which belong to the $SU(l)$ representation $D^{f_k}(u)$, the Hamiltonian $\hat{H}(k)$ takes the simple form $\hat{H}(k) = D^{f_m}(u)H(k)[D^{f_m}(u)]^+$ and is unitarily equivalent to $H(k)$, a property which is lacking for a generic $U \in U(N_k)$. For the parametrization Eq. (17), the matrices $H(k)$ are replaced by

$$\hat{H}_{v_m v'_m}(k) = \sum_{\kappa \kappa'} \langle m v_m | B_k(\kappa') | m v'_m \rangle W_k(\kappa) \Delta_{\kappa' \kappa}^k(U) . \quad (21)$$

Here $\Delta^k(U)$ denotes the matrix representation of $U \in U(N_k)$, carried by the matrices $\langle k v_k | B_k(\kappa) | k v'_k \rangle$,

$$\sum_{\tilde{v}_k \tilde{v}'_k} U_{v_k \tilde{v}_k} \langle k \tilde{v}_k | B_k(\kappa) | k \tilde{v}'_k \rangle (U^+)_{\tilde{v}'_k v'_k} = \sum_{\kappa'} \Delta_{\kappa' \kappa}^k(U) \langle k v_k | B_k(\kappa') | k v'_k \rangle . \quad (22)$$

This representation is unitary and real. It is the direct sum of two irreducible representations, the identity representation carried by the matrix of $B_k(0)$, and the $(N_k^2 - 1)$ -dimensional irreducible representation of the Young tableau $(10^{N_k^2-3}1)$ carried by the matrices of $B_k(\kappa)$ with $b > 0$. For $U = D^{f_k}(u)$, the matrices $\Delta^k(U)$ simplify to $\Delta_{\kappa' \kappa}^k(U) = \delta_{b b'} D_{w'_b w_b}^{g_b}(u)$. More details on the matrices $\Delta^k(U)$ are given in Appendix B. The invariance of the ensemble under $U(N_k)$ implies that whenever the integrals over $W_k(\kappa)$ are performed and the ensemble averages are expressed in terms of the matrix elements of the operators $B_k(\kappa)$, the resulting formulae must be invariant under the replacement of $B_k(\kappa)$ by

$$\hat{B}_k(\kappa) = \sum_{\kappa'} \Delta_{\kappa' \kappa}^k(U) B_k(\kappa') , \quad U \in U(N_k) . \quad (23)$$

As pointed out in ABRW, the properties of $\text{EGUE}(k)$ are closely related to those of the “dual” ensemble $\text{EGUE}(s) = \text{EGUE}(m - k)$. The dual ensemble describes m particles interacting by an s -particle GUE interaction $W(s)$. The Clebsch–Gordan coefficients $C_{f_k v_k f_s v_s}^{f_m v_m}$ and $C_{f_s v_s f_k v_k}^{f_m v_m}$ differ at most by a phase independent of the v ’s. Therefore, the Hamiltonian $H(s)$ of the dual ensemble can be written as

$$H_{v_m v'_m}(s) = \binom{m}{s} \sum_{v_k v_s v'_s} (C_{f_k v_k f_s v_s}^{f_m v_m})^* C_{f_k v_k f_s v'_s}^{f_m v'_m} W_{v_s v'_s}(s) , \quad s = m - k . \quad (24)$$

We use the link established by duality between $\text{EGUE}(k)$ and $\text{EGUE}(s)$ in Section 4.

4 The Matrix of Second Moments

In ABRW, the matrix of second moments plays a central role for the analysis of $\text{EGUE}(k)$. In the present Section, we address this matrix from the point of view of group theory.

The matrix elements $H_{v_m v'_m}(k)$ are Gaussian random variables. Therefore, all information necessary for the evaluation of averages over the ensemble $\text{EGUE}(k)$ is contained in the matrix of second moments,

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k) = \overline{H_{v_m v'_m}(k) H_{\tilde{v}_m \tilde{v}'_m}(k)} . \quad (25)$$

As shown by ABRW, central information on the properties of $\text{EGUE}(k)$ can be deduced if a “generalized eigenvalue expansion” for this matrix can be found. In particular, the application of Efetov’s supersymmetric averaging technique [9] to $\text{EGUE}(k)$ becomes possible. Such an expansion has the form

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k) = \sum_{\alpha} C_{v_m \tilde{v}'_m}^{\alpha} \Lambda^{\alpha} C_{\tilde{v}_m v'_m}^{\alpha} . \quad (26)$$

Here, the “eigenvalues” Λ^{α} must be positive, and the “eigenvectors” $C_{v_m \tilde{v}'_m}^{\alpha}$ should be Hermitean. In ABRW, the expansion (26) was constructed explicitly.

We show now that this expansion is closely related to the bodyness expansion of the Hamiltonian $H(s)$ of the dual ensemble $\text{EGUE}(s)$. Following ABRW, we note first that the second moments of the Hamiltonian matrix elements of the ensembles $\text{EGUE}(k)$ and $\text{EGUE}(s)$ satisfy the “duality” relation

$$\overline{N_k H_{v_m v'_m}(k) H_{\tilde{v}_m \tilde{v}'_m}(k)} = \overline{N_s H_{v_m \tilde{v}'_m}(s) H_{\tilde{v}_m v'_m}(s)} . \quad (27)$$

Here, the average on the left-hand side is over $\text{EGUE}(k)$, the average on the right-hand side is over $\text{EGUE}(s)$. The duality relation can be verified by expressing the matrix elements of $H(k)$ and $H(s)$ in terms of the coefficients of fractional parentage $C_{f_k v_k f_s v_s}^{f_m v_m}$ (Eqs. (15) and (24)), and taking the ensemble averages of the products of the interaction matrix elements $W_{v_k v'_k}(k)$ and $W_{v_s v'_s}(s)$. The duality relation is robust with respect to a truncation of Hilbert space: It remains valid when some of the basis states $|m v_m\rangle$ are excluded. However, this relation is obviously violated when the random k -particle interactions are modified in such a way that their ensembles lose their unitary symmetry so that Eq. (12) is not valid any more. An important

example of such a modification is the restriction of the interaction to the terms of the highest possible bodyness [10].

Another important observation relates to duality. The duality relation allows us to evaluate the second moments of $\text{EGUE}(k)$ in terms of the second moments of $\text{EGUE}(s)$. This yields the moments $\overline{H_{v_m v'_m}(k) H_{\tilde{v}_m \tilde{v}'_m}(k)}$ as bilinear forms of matrix elements of operators $B_s(\kappa)$. From the $U(N_s)$ symmetry of $\text{EGUE}(s)$, the resulting expressions are invariant under the replacement of $B_s(\kappa)$ by

$$\hat{B}_s(\kappa) = \sum_{\kappa'} \Delta_{\kappa' \kappa}^s(U) B_s(\kappa') \quad (28)$$

for any $U \in U(N_s)$. Here, $\Delta^s(U)$ denotes the transformation matrix defined analogously to Eq. (22). The same “dual” $U(N_s)$ invariance holds obviously for any ensemble average $\overline{F(H(k))}$ calculated from the second moments $\overline{H_{v_m v'_m}(k) H_{\tilde{v}_m \tilde{v}'_m}(k)}$ evaluated in this way. This concerns, in particular, the supersymmetric n -point generating functions of $\text{EGUE}(k)$.

We turn to the matrix of second moments $\overline{H_{v_m \tilde{v}'_m}(s) H_{\tilde{v}_m v'_m}(s)}$ of the Hamiltonian $H(s)$. We decompose the interaction $W(s)$ into a sum of terms with well-defined bodyness,

$$H_{v_m v'_m}(s) = \sum_{\kappa} \langle m v_m | B_s(\kappa) | m v'_m \rangle W_s(\kappa) . \quad (29)$$

We take the average of pairs of the random variables $W_s(\kappa)$ and express the matrix elements $\langle m v_m | B_s(\kappa) | m v'_m \rangle$ with the help of the $SU(l)$ Wigner-Eckart formula Eq. (9). We use the duality relation Eq. (27) and get

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k) = \overline{H_{v_m v'_m}(k) H_{\tilde{v}_m \tilde{v}'_m}(k)} = \sum_{b w_b}^s C_{v_m \tilde{v}'_m}^{b w_b} \Lambda^b(k) C_{\tilde{v}_m v'_m}^{b w_b} \quad (30)$$

where

$$\Lambda^b(k) = \frac{\lambda^2}{N_k} \langle m || B_s(b) || m \rangle^2 , \quad C_{v_m v'_m}^{b w_b} = C_{f_m v_m \bar{f}_m v'_m}^{g_b w_b} . \quad (31)$$

The upper limit at the summation symbol indicates that the summation over b is restricted to $b \leq s$. Equation (31) constitutes the eigenvalue expansion of the matrix $A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k)$, with $\Lambda^b(k) \geq 0$ the eigenvalues and $C^\kappa = (C_{v_m v'_m}^\kappa)$ the Hermitean eigenvectors. The eigenvalues $\Lambda^b(k)$ are labelled only by the bodyness quantum number b .

We demonstrate the dual $U(N_s)$ invariance of this spectral decomposition of $A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k)$ by writing Eq. (30) as (in the sequel we suppress the upper limit s at the summation symbol)

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k) = \frac{1}{N} \sum_{\kappa} V_{v_m \tilde{v}'_m}^{\kappa}(k) V_{\tilde{v}_m v'_m}^{\kappa}(k) . \quad (32)$$

Here $V^{\kappa}(k)$ stands for

$$V_{v_m v'_m}^{\kappa}(k) = \sqrt{N \Lambda^{\kappa}(k)} C_{v_m v'_m}^{\kappa} = \lambda \sqrt{N/N_k} \langle m v_m | B_s(\kappa) | m v'_m \rangle . \quad (33)$$

Under the $U(N_s)$ symmetry transformations Eq. (28), the matrices $V^{\kappa}(k)$ transform as

$$\hat{V}^{\kappa}(k) = \sum_{\kappa'} \Delta_{\kappa' \kappa}^s(U) V^{\kappa'}(k) , \quad U \in U(N_s) . \quad (34)$$

Since the matrix $\Delta^s(U)$ is unitary and real, the sum over κ appearing in Eq. (32) remains invariant under this transformation. This fact proves the invariance of $A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}(k)$.

The expansion (30) gives the matrix of the second moments in terms of the $SU(l)$ reduced matrix elements $\langle m || B_s(b) || m \rangle$ and in terms of the $SU(l)$ Clebsch-Gordan coefficients $C_{f_m v_m \bar{f}_m v'_m}^{g_b w_b}$. The reduced matrix elements of $B_s(\kappa)$ yield the eigenvalues and can be expressed in terms of the $SU(l)$ recoupling coefficients $\langle ((f_m \bar{f}_k) f_s f_k) f_m | (f_m (f_k \bar{f}_k) g_b) f_m \rangle$. However, as indicated by Eq. (10), these matrix elements can be evaluated even more directly by constructing suitable interactions of the required bodyness and particle rank, and by calculating the corresponding trace. We present this calculation in Appendix C. In the Fermionic case this yields

$$\Lambda^b(k) = \frac{\lambda^2}{N_k} \binom{m-b}{k} \binom{l-m+k-b}{k} , \quad (35)$$

whereas in the Bosonic case we get

$$\Lambda^b(k) = \frac{\lambda^2}{N_k} \binom{m-b}{k} \binom{l+m+b-1}{k} . \quad (36)$$

According to Eq. (27), the eigenvalue expansions of the matrices of second moments of the two dual ensembles $EGUE(k)$ and $EGUE(s)$ are related by

$$N_k \sum_{\kappa} C_{v_m \tilde{v}'_m}^{\kappa} \Lambda^b(k) C_{\tilde{v}_m v'_m}^{\kappa} = N_s \sum_{\kappa} C_{v_m v'_m}^{\kappa} \Lambda^b(s) C_{\tilde{v}_m \tilde{v}'_m}^{\kappa} . \quad (37)$$

When we compare these results with the eigenvalue expansion obtained in ABRW we see that, aside from normalization factors, the two expansions agree. Indeed, the eigenvalues found in ABRW agree with Eqs. (35) and (36), except for the factors λ^2/N_k which are missing in ABRW. This difference is caused by the fact that in ABRW, the second moment of the interaction matrix elements is normalized to unity, whereas in Eq. (12) we have used the standard GUE normalization for k interacting particles. We conclude that the eigenvalues of ABRW are identical to the squares of the reduced matrix elements $\langle m || B_s(b) || m \rangle$, and that the eigenvalues $C_{\mu\nu}^{sa}$ are, aside from a factor \sqrt{N} due to the difference in normalization, given by the Clebsch–Gordan coefficients $C_{v_m v'_m}^{bw_b}$. The factors N_k and N_s appearing on either side of the duality relations Eqs. (27) and (37) are likewise absent in ABRW. Again, this is due to the difference in normalization of the random variables.

5 Invariance under the Group $U(N)$

We have stressed in Section 3 that for the free GUE ensemble in N dimensions (or, equivalently, for $\text{EGUE}(m)$), the matrix of second moments is invariant under the transformation of the Hamiltonian by any unitary $N \times N$ matrix $U \in U(N)$, while for $k < m$, the matrix of the second moments of $\text{EGUE}(k)$ does not have this symmetry property. In the present section, we display the broken $U(N)$ symmetry of $\text{EGUE}(k)$ explicitly.

To determine the $U(N)$ –invariant part $A^{(0)}(k)$ of the matrix of second moments $A(k)$, we consider a transformation $U \in U(N)$ with $H(k) \rightarrow \hat{H}(k) = UH(k)U^\dagger$. Under this transformation, the components $C_{v_m v'_m}^\kappa$ of the eigenvector C^κ transform like the components of an $U(N)$ tensor with one covariant index and one contravariant index. Starting from Eq. (30), separating the $U(N)$ –invariant part of $C_{v_m \tilde{v}'_m}^\kappa C_{\tilde{v}_m v'_m}^\kappa$, and working out the sum over κ with the help of the duality relation Eq. (37), we find that the $U(N)$ –invariant part $A^{(0)}(k)$ has the form

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}^{(0)}(k) = \frac{N\Lambda^0(k)}{N^2 - 1} \left\{ \delta_{v_m \tilde{v}'_m} \delta_{\tilde{v}_m v'_m} \left(1 - S(k) \right) + \delta_{v_m v'_m} \delta_{\tilde{v}_m \tilde{v}'_m} N \left(S(k) - \frac{1}{N^2} \right) \right\}, \quad (38)$$

where

$$\Lambda^0(k) = \frac{1}{N} \overline{\langle H^2(k) \rangle_m}, \quad S(k) = \frac{1}{N} \frac{N_s \Lambda^0(s)}{N_k \Lambda^0(k)} = \frac{1}{N} \frac{\overline{\langle H(k) \rangle_m^2}}{\overline{\langle H^2(k) \rangle_m}}. \quad (39)$$

The $U(N)$ -invariant part of $A(k)$ is thus specified by the square $\Lambda^0(k)$ of the average spectral width and by the ratio $S(k)$. This ratio was introduced in ABRW as a measure of the ergodicity of the spectral centroids. We remark in parentheses that the converse is also true: The average spectral width $\sqrt{\Lambda^0(k)}$ and the ratio $S(k)$ are governed entirely by $A^{(0)}(k)$. There are no contributions either to $\Lambda^0(k)$ or to $S(k)$ arising from the $U(N)$ -non-invariant part of $A(k)$.

The result displayed in Eq. (38) is somewhat unexpected. Indeed, the $U(N)$ -invariant matrix $A^{(0)}(k)$ is the sum of two terms, $A^{(0)}(k) = A^G(k) + A^D(k)$. The first,

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}^G(k) = \frac{1}{N} \Lambda^G(k) \delta_{v_m \tilde{v}'_m} \delta_{\tilde{v}_m v'_m} \quad (40)$$

has the expected form: The Kronecker symbols carry the same indices as in the matrix of second moments of the free Gaussian unitary ensemble of $N \times N$ matrices or, equivalently, in $EGUE(m)$. The average spectral width Λ^G corresponding to the GUE described by this term is given by

$$\Lambda^G(k) = \Lambda^0(k) N^2 \frac{1 - S(k)}{N^2 - 1}. \quad (41)$$

The form of the second term,

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m}^D(k) = \Lambda^D(k) \delta_{v_m v'_m} \delta_{\tilde{v}_m \tilde{v}'_m}, \quad (42)$$

is unexpected. Indeed, the indices on the Kronecker deltas occur in the same way as in the matrix of second moments of the degenerate Gaussian ensemble of multiples of the $N \times N$ unit matrix or, equivalently, in $EGUE(0)$. The average spectral width corresponding to the degenerate Gaussian ensemble described by this term is given by

$$\Lambda^D(k) = \Lambda^0(k) \frac{N^2 S(k) - 1}{N^2 - 1}. \quad (43)$$

The ensemble EGUE(k) thus differs from the free GUE ensemble not only by the presence of the $U(N)$ -non-invariant part but also in the structure of the $U(N)$ -invariant part $A^{(0)}(k)$. This part contains, in addition to the expected term $A^G(k)$, also the term $A^D(k)$. The relative importance of the terms $A^G(k)$ and $A^D(k)$ is governed by the parameter $S(k)$. For fixed m and l , this parameter decreases with increasing k from the value unity at $k = 0$ to the value N^{-2} at $k = m$, with $S = N^{-1}$ at $k = m/2$. At $k = 0$, the GUE term $A^G(k)$ is absent since $\Lambda^G(0) = 0$. With increasing k , the width Λ^G increases, and for $k \geq m/2$ and large N , becomes dominant, $\Lambda^G(k) = \Lambda^0(k)$ up to terms of order $O(N^{-1})$. For finite $N \gg 1$ and small k , the $U(N)$ -invariant part $A^{(0)}(k)$ favors a Gaussian spectral shape and non-ergodic behavior, while for $k > m/2$, the favored spectral shape becomes semicircular and the spectral fluctuations, ergodic.

The $U(N)$ symmetry of EGUE(k) is obviously violated most strongly when k is close to $m/2$. A convenient measure for the degree of symmetry breaking is given by the ratio $P(k)$ of the norm of the $U(N)$ -invariant part $A^{(0)}(k)$ and of the norm of $A(k)$. From Eqs. (38) and (30) we find that $P(k)$ is given by

$$P(k) = \frac{\text{Tr}([A^{(0)}(k)]^+ A^{(0)}(k))}{\text{Tr}(A^+(k)A(k))} = \frac{1 - 2S(k) + N^2 S^2(k)}{(N^2 - 1)(R(k)/2)}. \quad (44)$$

Here $R(k)$ denotes the coefficient

$$R(k) = \frac{2 \sum_b^s M_b(\Lambda^b(k))^2}{(N \Lambda^0(k))^2} = \frac{\overline{\langle H^2(k) \rangle_m^2}}{(\overline{\langle H^2(k) \rangle_m})^2} - 1 \quad (45)$$

first introduced in ABRW as the measure of ergodicity of the spectral widths. The ratio $P(k)$ is symmetric about the point $k = m/2$ and attains its minimum there.

The results of the present Section cast new light on the most surprising result of ABRW. There it was found that for $k \ll m$, the spectral fluctuations of EGUE(k) differ markedly from Wigner–Dyson form and tend towards Poissonian behavior. While the present analysis does not yield direct information on this question, the fact that for $k \ll m$ the embedded ensemble EGUE(k) is dominated by the $U(N)$ -invariant term $A^D(k)$, lends additional plausibility to this result.

6 Symmetry Properties of the Supersymmetric Generating Functions

We show how symmetry arguments are involved in the use of the supersymmetry approach introduced by Efetov [9] and developed by Verbaarschot, Weidenmüller and Zirnbauer[11]. We use the same conventions as in Ref. [11]. We address first the one-point function and its saddle-point solution and then turn briefly to the generating functions of higher order.

The ensemble average of the one-point function $Z(E, H)$ of energy E is given by the graded integral

$$\overline{Z(E, H)} = \int d\mu(\psi) \exp \left\{ i\psi^+ (E_+ 1 \times 1 + J) \psi - \frac{1}{2} \overline{(\psi^+ (1 \times H) \psi)^2} \right\} \quad (46)$$

over the field ψ of $2N$ components $\psi_{\alpha v_m}$. The components with $\alpha = 0$ are ordinary complex variables, the components with $\alpha = 1$ anticommute. The matrices which appear in the exponent are written as direct products of the 2×2 matrices acting on the indices α and the $N \times N$ matrices acting on the indices v_m . The complex energy $E_+ = E + i\eta$ contains the infinitesimal term $\eta = 0_+$ introduced to assure convergence, and J is the source matrix $J_{\alpha v_m, \alpha' v'_m} = \delta_{\alpha\alpha'} (-)^{\alpha+1} \xi_{v_m v'_m}$. For notational simplicity, we suppress the index k throughout and write $H(k) = H$.

All information about the ensemble is contained in the matrix of second moments A . We first consider a toy model. We omit the $U(N)$ -non-invariant part of the matrix A . Then, A is given by the $U(N)$ -invariant part $A^{(0)} = A^G + A^D$ defined in Section 5. Making use of the transformation

$$\exp \left\{ -\frac{1}{2} \Lambda^D (\psi^+ \psi)^2 \right\} = \left(\frac{1}{2\pi \Lambda^D} \right)^{1/2} \int dt \exp \left\{ -\frac{1}{2\Lambda^D} (t^2 + 2it\Lambda^D \psi^+ \psi) \right\}, \quad (47)$$

we find that in this case, the function $\overline{Z(E, H)}$ simplifies to the convolution

$$\overline{Z^{(0)}(E)} = \left(\frac{1}{2\pi \Lambda^D} \right)^{1/2} \int dt \exp \left(-\frac{1}{2\Lambda^D} t^2 \right) \overline{Z^G(E - t, \Lambda^G)} \quad (48)$$

of the Gaussian of width $\sqrt{\Lambda^D}$ with the one-point function $\overline{Z^G(E, \Lambda^G)}$ of the free GUE in N dimensions with the average spectral width $\sqrt{\Lambda^G}$. The

average level density $\overline{\rho^{(0)}(E)}$ corresponding to $\overline{Z^{(0)}(E)}$ has the width $\sqrt{\Lambda^0}$ with $\Lambda^0 = \Lambda^G + \Lambda^D$. As for the kurtosis of the average level density, we follow the convention of ABRW: Their quantity $Q^{(0)}$ equals unity in case the spectral shape is Gaussian, and zero if the spectrum has semicircular shape. We find $Q^{(0)} = 1 - (\Lambda^G/\Lambda^0)^2$. Thus, $Q^{(0)}$ decreases from the value one at $k = 0$ to zero at $k = m$. For large N , the average level density is given by the convolution of a Gaussian of width $\sqrt{\Lambda^D}$ with the semicircular distribution of width $\sqrt{\Lambda^G}$. In general, the actual behavior of $\overline{Z(E, H)}$ is much more complex than that of this toy model. However, the toy model may perhaps be not far from the truth for $k \ll m$ and for k close to m . In both cases the non-invariant part of A is comparatively small.

We now address the one-point function in its full generality. With the help of the eigenvalue expansion of the matrix of second moments A and the Hubbard–Stratonovich transformation, the quartic term appearing in the exponent of Eq. (46) can be simplified and the integrals over ψ can be performed. This yields

$$\begin{aligned} \overline{Z(E, H)} = \int d\mu(\sigma) \exp \left\{ -\frac{1}{2} N \sum_{\kappa} \langle (\sigma^{\kappa})^2 \rangle \right. \\ \left. - \left\langle \ln \left(E_+ 1 \times 1 + J - \sum_{\kappa} \sigma^{\kappa} \times V^{\kappa} \right) \right\rangle \right\}, \end{aligned} \quad (49)$$

where $d\mu(\sigma)$ denotes the measure for integration over the graded 2×2 matrices σ^{κ} introduced in the Hubbard–Stratonovich transformation. We use $V^{\kappa} = \sqrt{N\Lambda^b} C^{\kappa}$. The angular brackets denote graded traces. Under a dual unitary transformation $U \in U(N_s)$, the matrices V^{κ} are replaced by the matrices \hat{V}^{κ} introduced in Eq. (34). Since $\Delta^s(U)$ are real unitary matrices, the transformation coefficients can be absorbed in the new integration variables $\hat{\sigma}^{\kappa} = \sum_{\kappa'} \Delta_{\kappa\kappa'}^s(U) \sigma^{\kappa'}$, with $\sum_{\kappa} (\hat{\sigma}^{\kappa})^2 = \sum_{\kappa} (\sigma^{\kappa})^2$. The integral thus remains invariant as required.

The same symmetry consideration limits the form of the saddle-point solution. Indeed, with $\sigma_{\text{sp}}^{\kappa}$ the saddle-point solution, the dual $U(N_s)$ invariance implies that the saddle-point approximation to the generating function

$$\overline{Z_{\text{sp}}(E, H)} = \exp \left\{ - \left\langle \ln \left(E_+ 1 \times 1 + J - \sum_{\kappa} \sigma_{\text{sp}}^{\kappa} \times V^{\kappa} \right) \right\rangle \right\} \quad (50)$$

must be invariant under the replacement of V^{κ} by \hat{V}^{κ} , Eq. (34), for any $U \in U(N_s)$. This condition can be fulfilled only when $\sigma_{\text{sp}}^{\kappa} = 0$ for all $b > 0$.

The saddle-point solution found in ABRW has just this form, with

$$\sigma_{\text{sp}}^\kappa = \delta_{b0} \sigma_0, \quad \sigma_0 = \left(\frac{E}{2\sqrt{\Lambda^0}} - i\sqrt{1 - \left(\frac{E}{2\sqrt{\Lambda^0}} \right)^2} \right) \cdot 1. \quad (51)$$

We stress that this saddle-point solution exists only by virtue of the fact that the basis states $|mv_m\rangle$ represent a complete set of states of the irreducible representation $D^{f_m}(u)$. This can be seen in the following way. As discussed in ABRW, the saddle-point equation has the form

$$N(\sigma_{\text{sp}}^\kappa)_{\alpha\alpha'} = \sum_{v_m v'_m} \left(E \cdot 1 \times 1 - \sum_{\kappa'} \sigma_{\text{sp}}^{\kappa'} \times V^{\kappa'} \right)_{\alpha v_m, \alpha' v'_m}^{-1} V_{v'_m v_m}^\kappa. \quad (52)$$

Substitution shows that $\sigma_{\text{sp}}^\kappa = \delta_{b0} \sigma_0$ is a solution only if the traces of the matrices V^κ vanish for all $b > 0$. These traces differ only by a factor from the traces $\langle B_m(\kappa) \rangle_m$. Moreover, only scalar quantities can have a nonzero trace in the complete representation space of an irreducible representation. Hence this condition is met. However, the exclusion of some of the states $|mv_m\rangle$ from the Hilbert space of the ensemble would make this argument invalid. The supersymmetry approach could then still be applied, but a saddle point would not exist generically.

For the generating functions of higher order, the symmetry properties of the ensemble manifest themselves in the same way. As shown in detail in ABRW, the two-point function can be written as the graded integral

$$\overline{Z(E + \omega, E - \omega, H)} = \int d\mu(\sigma) \exp \left\{ -\frac{1}{2} N \sum_{\kappa} \langle (\sigma^\kappa)^2 \rangle - \left\langle \ln \left((E \cdot 1 + \omega_+ L) \times 1 + J - \sum_{\kappa} \sigma^\kappa \times V^\kappa \right) \right\rangle \right\} \quad (53)$$

over the 4×4 graded matrices $\sigma^\kappa = (\sigma_{\alpha p, \alpha' p'}^\kappa)$. The doubling of dimensions of the graded matrices reflects the fact that we are dealing with two propagators, the retarded propagator of energy $E + \omega$ ($p = 1$), and the advanced propagator of energy $E - \omega$ ($p = 2$). The diagonal matrix L with $L_{\alpha p, \alpha' p'} = \delta_{\alpha\alpha'} \delta_{pp'} (-)^{p+1}$ distinguishes between the two cases. The matrix J contains the source parameters. As discussed thoroughly in ABRW, the

integral is dominated by the saddle-point manifold

$$\sigma_{\text{sp}}^\kappa = \delta_{b0} T^{-1} \sigma_0 T, \quad \sigma_0 = \frac{E}{2\sqrt{\Lambda^0}} \cdot 1 - i \sqrt{1 - \left(\frac{E}{2\sqrt{\Lambda^0}} \right)^2} \cdot L. \quad (54)$$

Here, $T = (T_{\alpha p, \alpha' p'})$ denote the 4×4 graded matrices belonging to the coset space $U(1, 1/2)/[U(1/1) \times U(1/1)]$. The integral as well as the saddle-point approximation are manifestly invariant with respect to the dual symmetry transformations $U \in U(N_s)$, Eq. (34). Saddle-point manifolds of the same structure dominate also the graded integrals which represent the generating functions of higher order.

7 Discussion and Conclusions

In this paper, we have shown that some of the central results of ABRW can be deduced with the help of symmetry considerations. This fact explains why analytical progress in understanding the properties of $\text{EGUE}(k)$ has been possible at all. In addition, our work offers a deeper insight into the structure of $\text{EGUE}(k)$. We mention, in particular, the decomposition in Section 5 of the matrix of second moments into a part which is invariant under $U \in U(N)$ and another which is not.

We mention the following generalizations of our work.

(i) The theory developed in Sections 3 - 6 for $\text{EGUE}(k)$ is based on the $SU(l)$ expansion formula Eq. (15). We emphasize that the approach developed in ABRW and in the present paper is not restricted to this case but is much more general. Indeed, it applies likewise to ensembles defined by the same formula but with the $SU(l)$ Clebsch-Gordan coefficients replaced by the Clebsch-Gordan coefficients of another symmetry group G . We will now discuss this case in order to illuminate the algebraic structure of the approach used in the present paper.

We introduce a generalized ensemble by considering two independent systems labelled $j = k$ and $j = s$ whose basic states $|f_j v_j\rangle$ transform according to the irreducible representations $D^{f_j}(g)$ of dimension N_j of the group G . We assume for definiteness that the group G , $g \in G$, is a compact simple Lie group; the irreducible representations of G will be labelled by their highest weights f , their basis states by the running indices v . We assume that

a non-trivial interaction $W(k)$ of GUE type occurs only in system k , with $W_{v_k v'_k}(k) = \langle f_k v_k | W(k) | f_k v'_k \rangle$ Gaussian and distributed according to Eq. (12). The embedding of this interaction into a space of different dimension is accomplished by projecting the product states $|f_k v_k \rangle |f_s v_s \rangle$ onto the subspace of states which transform according to the irreducible representation $D^{f_m}(g)$ contained in the direct product $D^{f_k}(g) \times D^{f_s}(g)$. We denote the associated projection operator by $I(f_m)$. The ensemble is then defined in terms of the Hamiltonian

$$H(k) = I(f_m)W(k)I(f_m) . \quad (55)$$

Eq. (55) comprises the essence of the group-theoretical extension of the idea of an embedded ensemble. This extension is independent of the existence of Fermions and Bosons. It relies only on group-theoretical concepts.

Using the standard composition formula

$$|r_m f_m v_m \rangle = \sum_{v_k v_s} |f_k v_k \rangle |f_s v_s \rangle C_{f_k v_k f_s v_s}^{r_m f_m v_m} , \quad (56)$$

we find for the matrix elements of $H(k)$

$$H_{r_m v_m, r'_m v'_m}(k) = \sum_{v_k v'_k v_s} (C_{f_k v_k f_s v_s}^{r_m f_m v_m})^* C_{f_k v'_k f_s v_s}^{r'_m f_m v'_m} W_{v_k v'_k}(k) . \quad (57)$$

Here $C_{f_k v_k f_s v_s}^{r_m f_m v_m}$ denotes the Clebsch–Gordan coefficient of the group G for the coupling $(f_k f_s) f_m$. The multiplicity index r_m distinguishes different $D^{f_m}(g)$ of the same highest weight f_m which may appear in the reduction of $D^{f_k}(g) \times D^{f_s}(g)$. The dimension of the matrices $H(k)$ is $N = \mu N_m$, where N_m denotes the dimension of $D^{f_m}(g)$ and μ the multiplicity with which this representation appears in the reduction. By construction, the ensemble is invariant under unitary transformations $U \in \text{U}(N_k)$ of dimension N_k of the interaction matrices $W_{v_k v'_k}(k)$.

In analogy to Eq. (24), we can introduce the dual ensemble

$$H(s) = I(f_m)W(s)I(f_m) , \quad (58)$$

with the random GUE interaction $W(s)$ acting now on the second system s . Eqs. (57) and (12) and analogous equations valid for the Hamiltonian $H(s)$ and the interaction $W(s)$ imply for the matrices of second moments $A(k)$ and $A(s)$ of the two dual ensembles the duality relation (see Eq. (27))

$$\overline{N_k H_{r_m v_m, r'_m v'_m}(k) H_{\tilde{r}_m \tilde{v}_m, \tilde{r}'_m \tilde{v}'_m}(k)} = \overline{N_s H_{r_m v_m, \tilde{r}'_m \tilde{v}'_m}(s) H_{\tilde{r}_m \tilde{v}_m, r'_m v'_m}(s)} . \quad (59)$$

To derive the eigenvalue expansion of the matrix of second moments, we expand the interaction $W(s)$ in terms of a complete set of N_s^2 basic interactions $B_s(\kappa) = B_s^+(\kappa)$ normalized according to $\langle B_s(\kappa)B_s(\kappa') \rangle_s = \delta_{\kappa\kappa'}$, with $B_s(0)$ denoting again a normalized multiple of the unit operator. Here, $\langle O_s \rangle_s$ denotes the trace of O_s in the Hilbert space of states $|f_s v_s\rangle$ of the second system. This yields

$$W(s) = \sum_{\kappa} B_s(\kappa) W_s(\kappa), \quad \overline{W_s(\kappa)W_s(\kappa')} = \frac{\lambda^2}{N_s} \delta_{\kappa\kappa'}. \quad (60)$$

Using this expansion and proceeding as in Section 4, we find the eigenvalue expansion

$$\begin{aligned} A_{r_m v_m \tilde{r}'_m \tilde{v}'_m, \tilde{r}_m \tilde{v}_m r'_m v'_m}(k) &= \overline{H_{r_m v_m, r'_m v'_m}(k) H_{\tilde{r}_m \tilde{v}_m, \tilde{r}'_m \tilde{v}'_m}(k)} \\ &= \frac{\lambda^2}{N_k} \sum_{\kappa} \langle r_m f_m v_m | B_s(\kappa) | \tilde{r}'_m f_m \tilde{v}'_m \rangle \langle \tilde{r}_m f_m \tilde{v}_m | B_s(\kappa) | r'_m f_m v'_m \rangle. \end{aligned} \quad (61)$$

Basic properties of the ensemble can be derived from this formula. The eigenvalue expansion can eventually be simplified by adapting the choice of the operators $B_s(\kappa)$ to the group chain $U(N_s) \supset D^{f_s}$. We show in Appendix D that the traces $\langle B_s(\kappa) \rangle_m$ of the operators $B_s(\kappa)$ in the Hilbert space of the composite system vanish for $\kappa \neq 0$. This makes it possible to use the supersymmetry approach in a meaningful way. The graded integrals which represent the n -point functions of the ensemble are dominated by the saddle points and/or saddle-point manifolds analogous to those discussed in the preceding section.

(ii) We may consider $\text{EGUE}(k)$ as a member of a family of ensembles obtained by a modification of the GUE, the free Gaussian unitary ensemble in N dimensions. Eq. (17) shows that the GUE in N dimensions, $\text{GUE} = \text{EGUE}(m)$, may be written as

$$H_{v_m v'_m} = \sum_{bw_b} \langle m v_m | B_m(bw_b) | m v'_m \rangle W(bw_b), \quad (62)$$

with the sum over b running over all $b = 0, \dots, m$ and all corresponding w_b , and with $W(bw_b)$ Gaussian distributed with mean value zero and second moment

$$\overline{W(bw_b)W(b'w'_{b'})} = \frac{\lambda^2}{N} \delta_{bb'} \delta_{w_b w'_{b'}}. \quad (63)$$

When we restrict the sum over b to b not larger than k and renormalize the matrix elements by the b -dependent factors $K(b) = \sqrt{N/N_k} \langle m | B_k(b) | m \rangle$, the free Gaussian ensemble turns into EGUE(k). Other restrictions of the sum over b and other renormalizations $K(b)$ of the matrix elements will create other “modified GUE ensembles” of $N \times N$ matrices

$$\mathcal{H}_{v_m v'_m} = \sum'_{bw_b} \langle m v_m | B_m(b w_b) | m v'_m \rangle K(b) W(b w_b) . \quad (64)$$

The prime at the summation symbol indicates the restriction of the sum over b . By construction, the resulting modified ensembles are all invariant under the $SU(l)$ transformation $\mathcal{H} \rightarrow D^{f_m}(u) \mathcal{H} [D^{f_m}(u)]^+$. On expressing the matrix elements $\langle m v_m | B_m(b w_b) | m v'_m \rangle$ in terms of $SU(l)$ Clebsch-Gordan coefficients and performing the appropriate $SU(l)$ recoupling, we find for the matrix of moments of the modified ensemble the eigenvalue expansion

$$A_{v_m \tilde{v}'_m, \tilde{v}_m v'_m} = \overline{\mathcal{H}_{v_m v'_m}}_{\tilde{v}_m \tilde{v}'_m} = \sum_{bw_b} C_{v_m \tilde{v}'_m}^{g_b w_b} \Lambda^b C_{\tilde{v}_m v'_m}^{g_b w_b} . \quad (65)$$

Here, $C_{v_m \tilde{v}'_m}^{g_b w_b}$ denote the $SU(l)$ Clebsch-Gordan coefficients Eq. (31), and Λ^b the eigenvalues

$$\Lambda^b = \frac{\lambda^2}{N} \sum'_{\tilde{b}} \sqrt{\frac{M_{\tilde{b}}}{M_b}} K^2(\tilde{b}) \langle ((f_m \bar{f}_m) g_b (f_m \bar{f}_m) g_b) 0 | ((f_m \bar{f}_m) g_{\tilde{b}} (f_m \bar{f}_m) g_{\tilde{b}}) 0 \rangle . \quad (66)$$

The angular bracket expression stands for the $SU(l)$ recoupling coefficient for the indicated change of the coupling scheme, with 0 denoting the highest weight of the identity representation. For Λ^b positive, the properties of the modified ensemble thus can be studied by applying Efetov’s supersymmetric averaging technique in the same way as for EGUE(k).

The cases mentioned under (i) and (ii) are very instructive. They show that our group-theoretical approach to EGUE(k) is very general. Moreover, the results obtained for EGUE(k) are typical for a wide class of embedded ensembles defined in terms of a general compact group G .

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A The composition formula for $A^+(mv_m)$

The single-particle creation operators a_i^+ transform according to the $SU(l)$ representation $D^{f_1}(u)$. The reduction of the direct product $D^{f_{j-1}}(u) \times D^{f_1}(u)$ contains the representation $D^{f_j}(u)$ just once. Therefore, the creation operators $A^+(mv_m)$ can be written as

$$A^+(mv_m) = \frac{1}{\sqrt{m!}} \sum_{i_1 \dots i_m} a_{i_1}^+ \dots a_{i_m}^+ C_{i_1 \dots i_m}^{f_m v_m}, \quad (67)$$

where $C_{i_1 \dots i_m}^{f_m v_m}$ stands for the coupling coefficient

$$C_{i_1 \dots i_m}^{f_m v_m} = \sum_{v_2 v_3 \dots v_{m-1}} C_{f_1 i_1 f_1 i_2}^{f_2 v_2} C_{f_2 v_2 f_1 i_3}^{f_3 v_3} \dots C_{f_{m-1} v_{m-1} f_1 i_m}^{f_m v_m}. \quad (68)$$

This coefficient is given in terms of products of the $SU(l)$ Clebsch–Gordan coefficients $C_{f_{j-1} v_{j-1} f_1 i_j}^{f_j v_j}$ for the coupling $(f_{j-1} f_1) f_j$, with $j = 2, \dots, m$. In the Fermionic (Bosonic) case, the coefficients $C_{i_1 \dots i_m}^{f_m v_m}$ are totally antisymmetric (symmetric) functions of i_1, \dots, i_m . The factor $(m!)^{-1/2}$ takes care of the proper normalization $\langle |A(mv_m) A^+(mv_m)| \rangle = \langle mv_m | mv_m \rangle = 1$: The total number of contractions contributing to the norm is equal to $m!$. This statement applies equally to Fermions and to Bosons.

Indicating the coupling scheme explicitly, we can rewrite Eq. (67) in the symbolic form

$$A^+(mv_m) = \frac{1}{\sqrt{m!}} ((a^+ a^+)^{f_2} \dots a^+)^{f_m}_{v_m}. \quad (69)$$

Modifying the coupling scheme we find

$$((a^+ a^+)^{f_2} \dots a^+)^{f_m}_{v_m} = \sum_{v_k v_s} ((a^+ a^+)^{f_2} \dots a^+)^{f_k}_{v_k} ((a^+ a^+)^{f_2} \dots a^+)^{f_s}_{v_s} C_{f_k v_k f_s v_s}^{f_m v_m}, \quad (70)$$

where $C_{f_k v_k f_s v_s}^{f_m v_m}$ are the $SU(l)$ Clebsch–Gordan coefficients for the coupling $(f_k f_s) f_m$. Substituting this relation into Eq. (69) and introducing the creation operators $A^+(kv_k)$ and $A^+(sv_s)$, we obtain

$$A^+(mv_m) = \sqrt{\frac{k! s!}{m!}} \sum_{v_k v_s} A^+(kv_k) A^+(sv_s) C_{f_k v_k f_s v_s}^{f_m v_m}. \quad (71)$$

This is the composition formula which underlies Eq. (3).

Taking into account that, due to the anticommutativity (commutativity) of the Fermionic (Bosonic) operators a_i^+ , the products $A^+(kv_k)A^+(sv_s)$ can be coupled only to operators belonging to the irreducible tableaux f_m , i.e.,

$$\sum_{v_k v_s} A^+(kv_k)A^+(sv_s)C_{f_k v_k f_s v_s}^{fv} = 0 \quad \text{for } f \neq f_m, \quad (72)$$

and making use of orthonormality of Clebsch-Gordan coefficients, we find that the composition formula can be inverted,

$$A^+(kv_k)A^+(sv_s) = \left(\frac{m}{k}\right)^{1/2} \sum_{v_m} A^+(mv_m)(C_{f_k v_k f_s v_s}^{f_m v_m})^*. \quad (73)$$

From this equation it follows that the matrix elements of the creation operators $A^+(kv_k)$ have the form

$$\langle mv_m | A^+(kv_k) | sv_s \rangle = \left(\frac{m}{k}\right)^{1/2} (C_{f_k v_k f_s v_s}^{f_m v_m})^*. \quad (74)$$

Substituting this equation in

$$\langle mv_m | W(k) | mv'_m \rangle = \sum_{v_k v'_k v_s} \langle mv_m | A^+(kv_k) | sv_s \rangle \langle sv_s | A(kv'_k) | mv'_m \rangle W_{v_k v'_k}(k) \quad (75)$$

yields the formula Eq. (15).

B The matrices $\Delta^k(U)$

Multiplying Eq. (22) by $\langle kv'_k | B_k(\kappa'') | kv_k \rangle$ and summing it over v_k and v'_k with the help of the orthonormality relation (8), we arrive at the following explicit expression for $\Delta_{\kappa'\kappa}^k(U)$:

$$\Delta_{\kappa'\kappa}^k(U) = \sum_{v_k \tilde{v}_k v'_k \tilde{v}'_k} U_{v_k \tilde{v}_k} \langle k \tilde{v}_k | B_k(\kappa) | k \tilde{v}'_k \rangle (U^+)_{\tilde{v}'_k v'_k} \langle kv'_k | B_k(\kappa') | kv_k \rangle. \quad (76)$$

We substitute $\langle kv_k | B_k(0) | kv'_k \rangle = \delta_{v_k v'_k} (N_k)^{-1/2}$, use Eq. (76), and find that

$$\Delta_{0\kappa}^k(U) = \Delta_{\kappa 0}^k(U) = \delta_{\kappa 0}. \quad (77)$$

The representation $\Delta^k(U)$ thus has a fully reduced form: Under the unitary transformations $U \in U(N_k)$, the matrix of $B_k(0)$ remains invariant whereas the matrices of the $B_k(\kappa)$ with $b > 0$ transform like the components of an irreducible $U(N_k)$ tensor of dimension $N_k^2 - 1$ and of Young tableau $(10^{N_k-3}1)$. For $U = D^{f_k}(u)$, from Eq. (6) it follows that

$$\begin{aligned} \sum_{\tilde{v}_k \tilde{v}'_k} D_{v_k \tilde{v}_k}^{f_k}(u) \langle k \tilde{v}_k | B_k(bw_b) | k \tilde{v}'_k \rangle ([D^{f_k}(u)]^+)^{\tilde{v}'_k v'_k} \\ = \sum_{w'_b} D_{w'_b w_b}^{g_b}(u) \langle k v_k | B_k(bw'_b) | k v'_k \rangle . \end{aligned} \quad (78)$$

For $U = D^{f_k}(u)$, the matrices $\Delta^k(U)$ simplify to $\Delta_{\kappa' \kappa}^k(U) = \delta_{bb'} D_{w'_b w_b}^{g_b}(u)$.

C The reduced matrix elements $\langle m || B_s(b) || m \rangle$

To calculate the squares of the reduced matrix elements, we apply the approach by Mon and French [5] (cf. also Refs. [12, 13]). At the end of Section 2 it was shown that the value of $\langle m || B_s(b) || m \rangle^2$ can be obtained from the norm $\langle S_s^+(b) S_s(b) \rangle_m$ of a suitable s -particle operator $S_s(b)$ of bodyness b . Any such operator can be written as the product

$$S_s(b) = \binom{\hat{N} - b}{s - b} S_b(b) \quad (79)$$

of the $SU(l)$ invariant polynomial of the particle number operator \hat{N} given by the first factor and of a b -particle operator of bodyness b denoted by $S_b(b)$. The form of the polynomial is uniquely determined by the particle ranks of $S_s(b)$ and $S_b(b)$: Since $S_s(b)$ is an s -particle operator, the polynomial has to vanish when acting on the states of $t = b, \dots, s - 1$ particles. This reduces the choice of $S_s(b)$ to the choice of $S_b(b)$. The b -particle operators of bodyness b change the state of b particles and cannot be simulated by the operators of lower particle rank. For Fermions, a suitable choice then is $S_b(b) = \xi a_1^+ \dots a_b^+ a_{b+1} \dots a_{2b}$ while for Bosons, we use $S_b(b) = \xi (a_1^+)^b (a_2)^b / b!$. Here, ξ denotes a normalization factor. This yields

$$\langle S_b^+(b) S_b(b) \rangle_m = \xi^2 \binom{l - 2b}{m - b}, \quad \langle S_b^+(b) S_b(b) \rangle_m = \xi^2 \binom{l + m + b - 1}{m - b} \quad (80)$$

for Fermions and Bosons, respectively. We substitute this result in

$$\langle S_s^+(b)S_s(b) \rangle_m = \binom{m-b}{s-b}^2 \langle S_b^+(b)S_b(b) \rangle_m \quad (81)$$

and fix the value of ξ from the normalization condition $\langle S_s^+(b)S_s(b) \rangle_s = 1$. This yields

$$\begin{aligned} \langle m || B_s(b) || m \rangle^2 &= \langle S_s^+(b)S_s(b) \rangle_m \\ &= \binom{m-b}{k} \binom{l-m+k-b}{k} \quad (\text{Fermions}) \\ &= \binom{m-b}{k} \binom{l+m+b-1}{k} \quad (\text{Bosons}) , \end{aligned} \quad (82)$$

in keeping with Eqs. (35) and (36).

D The traces $\langle B_s(\kappa) \rangle_m$

Using the composition formula (56) we find that

$$\langle B_s(\kappa) \rangle_m = \sum_{r_m v_m v_k v_s v'_s} (C_{f_k v_k f_s v_s}^{r_m f_m v_m})^* C_{f_k v_k f_s v'_s}^{r_m f_m v_m} \langle f_s v_s | B_s(\kappa) | f_s v'_s \rangle . \quad (83)$$

We express the Clebsch–Gordan coefficients with the help of the symmetry relation (cf. Refs. [6, 14])

$$C_{f_k v_k f_s v_s}^{r_m f_m v_m} = \sqrt{\frac{N_m}{N_s}} \sum_{\bar{v}_k r_s} (C_{\bar{f}_k \bar{v}_k f_m v_m}^{r_s f_s v_s})^* U_{v_k \bar{v}_k}^{(f_k)} U_{r_s r_m}^{(f_k f_s f_m)} . \quad (84)$$

Here, \bar{f}_k denotes the highest weight of the irreducible representation conjugate to $D^{f_k}(g)$; \bar{v}_k the basis state of this representation; $C_{\bar{f}_k \bar{v}_k f_m v_m}^{r_s f_s v_s}$ the Clebsch–Gordan coefficient for the coupling $(\bar{f}_k f_m) f_s$; r_s the multiplicity index for this coupling; $U_{v_k \bar{v}_k}^{(f_k)}$ and $U_{r_s r_m}^{(f_k f_s f_m)}$ the unitary matrices

$$U_{v_k \bar{v}_k}^{(f_k)} = \sqrt{N_k} C_{f_k v_k \bar{f}_k \bar{v}_k}^{00} , \quad (85)$$

$$U_{r_s r_m}^{(f_k f_s f_m)} = \sqrt{\frac{1}{N_s N_m}} \sum_{v_k \bar{v}_k v_s v_m} C_{\bar{f}_k \bar{v}_k f_m v_m}^{r_s f_s v_s} C_{f_k v_k f_s v_s}^{r_m f_m v_m} (U_{v_k \bar{v}_k}^{(f_k)})^* . \quad (86)$$

In Eq. (85), (00) denotes the (fv) labels of the identity representation. Substituting Eq. (84) and performing the sum yields

$$\langle B_s(\kappa) \rangle_m = \frac{N}{N_s} \langle B_s(\kappa) \rangle_s = \delta_{\kappa 0} \frac{N}{\sqrt{N_s}} . \quad (87)$$

The traces are thus nonzero only for $\kappa = 0$.

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